

Zeta Functional Analysis

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Abstract

We intimate deeper connections between the Riemann zeta and gamma functions than often reported and further derive a new formula for expressing the value of $\zeta(2n+1)$ in terms of zeta at other fractional points. This paper also establishes and presents new expository notes and perspectives on zeta function theory and functional analysis. In addition, a new fundamental result, in form of a new function called omega $\Omega(s)$, is introduced to analytic number theory for the first time. This new function together with some of its most fundamental properties and other related identities are here disclosed and presented as a new approach to the analysis of sums of generalised harmonic series, related alternating series and polygamma functions associated with Riemann zeta function.

1 Introduction

The Riemann zeta function is defined by the generalised harmonic series

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \quad (1)$$

where $s = \sigma + it$ and $\sigma > 1$. In his 1859 paper [1], Riemann introduced the functional equation

$$\frac{\zeta(s)\Gamma(\frac{s}{2})}{\pi^{\frac{s}{2}}} = \frac{\zeta(1-s)\Gamma(\frac{1-s}{2})}{\pi^{\frac{1-s}{2}}} \quad (2)$$

which suggests replacing the value of s by $1-s$ without changing the result of the outcome after substitution. Our imagination is tickled by this property to investigate

$$\log\left(\frac{\zeta(s)\Gamma(\frac{s}{2})}{\pi^{\frac{s}{2}}}\right) = \log\left(\frac{\zeta(1-s)\Gamma(\frac{1-s}{2})}{\pi^{\frac{1-s}{2}}}\right) \quad (3)$$

hoping that we might find an elementary expression for the value of $\Gamma(1-s)$. Observe

$$\log\left(\frac{\zeta(s)}{\zeta(1-s)}\right) = \log\left(\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\right) + (s - \frac{1}{2})\log(\pi) = \log\left(\frac{(2\pi)^s}{\pi} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\right), \quad (4)$$

which implies

$$\log\left(\frac{\zeta(s)}{\zeta(1-s)}\right) = \log\left(\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\right) + (s - \frac{1}{2})\log(\pi) = s\log(2\pi) - \log(\pi) + \log\left(\sin\left(\frac{\pi s}{2}\right)\right) + \log(\Gamma(1-s)). \quad (5)$$

Without hesitation, we are induced to introduce $1-s$ in place of s into the last equation to produce:

$$\log\left(\frac{\zeta(1-s)}{\zeta(s)}\right) = \log\left(\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}\right) + (\frac{1}{2} - s)\log(\pi) = (1-s)\log(2\pi) - \log(\pi) + \log\left(\cos\left(\frac{\pi s}{2}\right)\right) + \log(\Gamma(s)), \quad (6)$$

similar to Riemann's idea. We digress from the original motivation of finding an expression for $\Gamma(1-s)$ to envision many great uses for these simple formulae created in 5 and 6, taking on great feats such as the challenge of finding a new formula for $\zeta(2n+1)$ resulting in a new development and progress towards tackling this long standing open problem in number theory.

2 Polygamma function and Riemann zeta at odd integers

Lemma 2.1. Assuming $n \geq 1$ is an integer number,

$$\frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\Gamma(s))) = \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\frac{\zeta(1-s)}{\zeta(s)})) - \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\cos(\frac{\pi s}{2}))) \quad (7)$$

Proof. According to 6

$$\frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\Gamma(s))) + \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\cos(\frac{\pi s}{2}))) = \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\frac{\zeta(1-s)}{\zeta(s)})). \quad (8)$$

□

Lemma 2.2. Assuming $n \geq 1$ is an integer number,

$$\frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\Gamma(\frac{s}{2}))) - \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\Gamma(\frac{1-s}{2}))) = \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\frac{\zeta(1-s)}{\zeta(s)})). \quad (9)$$

Proof. Again, this results as a direct consequence of 6. □

Lemma 2.3. The following identities are valid:

$$\frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\Gamma(s))) = \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\Gamma(\frac{s}{2}))) - \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\Gamma(\frac{1-s}{2}))) - \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\cos(\frac{\pi s}{2}))); \quad (10)$$

and

$$\frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\Gamma(1-s))) = -\frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\Gamma(\frac{s}{2}))) + \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\Gamma(\frac{1-s}{2}))) - \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\sin(\frac{\pi s}{2}))). \quad (11)$$

Proof. Again, this results as a direct consequence of 6. □

Theorem 2.4. Assuming n is an integer number, the following is a formula expressing the value of $\zeta(2n+1)$.

$$(-1)^{2n+1}(2^{2n+1}-1)\zeta(2n+1)\Gamma(2n+1) = \psi^{(2n)}(\frac{1}{2}) = \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\frac{\zeta(1-s)}{\zeta(s)})) \Big|_{s \rightarrow \frac{1}{2}} - \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\cos(\frac{\pi s}{2}))) \Big|_{s \rightarrow \frac{1}{2}} \quad (12)$$

Proof.

$$\frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\Gamma(s))) \Big|_{s \rightarrow \frac{1}{2}} = \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\frac{\zeta(1-s)}{\zeta(s)})) \Big|_{s \rightarrow \frac{1}{2}} - \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\cos(\frac{\pi s}{2}))) \Big|_{s \rightarrow \frac{1}{2}} \quad (13)$$

□

According to K.S. Kölbig [2], special values of polygamma function [3, 4] defined as $\psi^{(s-1)}(x) = \frac{d^{s-1}}{dx^{s-1}}\psi(x) = \frac{d^s}{dx^s} \ln \Gamma(x)$ may be combined to compute (or compose) the value of $\zeta(2n+1)$, e.g. $\psi^{(k)}(\frac{1}{4}) + \psi^{(k)}(\frac{3}{4})$ as in the case of:

$$\zeta(n) = (-1)^n \cdot \frac{(\psi^{(n-1)}(\frac{1}{4}) + \psi^{(n-1)}(\frac{3}{4}))}{2^n \cdot (2^n - 1)} \cdot \frac{1}{\Gamma(n)} \quad (14)$$

Theorem 2.5. *Let n be an integer number, then*

$$-\zeta(2n+1) = \frac{(\psi^{(2n)}(\frac{1}{4}) + \psi^{(2n)}(\frac{3}{4}))}{2^{2n+1}(2^{2n+1}-1)} \cdot \frac{1}{\Gamma(2n+1)} = \frac{2 \frac{d^{(2n+1)}}{ds^{(2n+1)}} \left(\log\left(\frac{\zeta(1-s)}{\zeta(s)}\right) \right) \Big|_{s \rightarrow \frac{1}{4}} + \frac{d^{(2n+1)}}{ds^{(2n+1)}} \left(\log\left(\tan\left(\frac{\pi s}{2}\right)\right) \right) \Big|_{s \rightarrow \frac{1}{4}}}{2^{2n+1}(2^{2n+1}-1)\Gamma(2n+1)} \quad (15)$$

Proof. Notice that

$$\psi^{(2n)}\left(\frac{1}{4}\right) = \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\Gamma(s))) \Big|_{s \rightarrow \frac{1}{4}} = \frac{d^{(2n+1)}}{ds^{(2n+1)}}\left(\log\left(\frac{\zeta(1-s)}{\zeta(s)}\right)\right) \Big|_{s \rightarrow \frac{1}{4}} - \frac{d^{(2n+1)}}{ds^{(2n+1)}}\left(\log\left(\cos\left(\frac{\pi s}{2}\right)\right)\right) \Big|_{s \rightarrow \frac{1}{4}} \quad (16)$$

$$\begin{aligned} \psi^{(2n)}\left(\frac{3}{4}\right) &= \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\Gamma(s))) \Big|_{s \rightarrow \frac{3}{4}} = \frac{d^{(2n+1)}}{ds^{(2n+1)}}\left(\log\left(\frac{\zeta(1-s)}{\zeta(s)}\right)\right) \Big|_{s \rightarrow \frac{3}{4}} - \frac{d^{(2n+1)}}{ds^{(2n+1)}}\left(\log\left(\cos\left(\frac{\pi s}{2}\right)\right)\right) \Big|_{s \rightarrow \frac{3}{4}} \\ &= \frac{d^{(2n+1)}}{ds^{(2n+1)}}\left(\log\left(\frac{\zeta(1-s)}{\zeta(s)}\right)\right) \Big|_{s \rightarrow \frac{1}{4}} + \frac{d^{(2n+1)}}{ds^{(2n+1)}}\left(\log\left(\sin\left(\frac{\pi s}{2}\right)\right)\right) \Big|_{s \rightarrow \frac{1}{4}} \end{aligned} \quad (17)$$

Therefore

$$\psi^{(2n)}\left(\frac{1}{4}\right) + \psi^{(2n)}\left(\frac{3}{4}\right) = 2 \frac{d^{(2n+1)}}{ds^{(2n+1)}}\left(\log\left(\frac{\zeta(1-s)}{\zeta(s)}\right)\right) \Big|_{s \rightarrow \frac{1}{4}} + \frac{d^{(2n+1)}}{ds^{(2n+1)}}\left(\log\left(\tan\left(\frac{\pi s}{2}\right)\right)\right) \Big|_{s \rightarrow \frac{1}{4}}. \quad (18)$$

□

Clearly the following theorems are valid and do not require explicit proofs.

Theorem 2.6. *In general,*

$$\begin{aligned} \psi^{(2n)}(s) + \psi^{(2n)}(1-s) &= 2 \frac{d^{(2n+1)}}{ds^{(2n+1)}}\left(\log\left(\frac{\zeta(1-s)}{\zeta(s)}\right)\right) + \frac{d^{(2n+1)}}{ds^{(2n+1)}}\left(\log\left(\tan\left(\frac{\pi s}{2}\right)\right)\right) \\ &= -2 \frac{d^{(2n+1)}}{ds^{(2n+1)}}\left(\log\left(\frac{\zeta(s)}{\zeta(1-s)}\right)\right) - \frac{d^{(2n+1)}}{ds^{(2n+1)}}\left(\log\left(\cot\left(\frac{\pi s}{2}\right)\right)\right), \end{aligned} \quad (19)$$

where $s \notin \{0, 1\}$.

Theorem 2.7. *In general,*

$$\psi^{(2n)}(s) - \psi^{(2n)}(1-s) = -\frac{d^{(2n+1)}}{ds^{(2n+1)}}\left(\log\left(\cos\left(\frac{\pi s}{2}\right) \sin\left(\frac{\pi s}{2}\right)\right)\right) \quad (20)$$

where $s \notin \{0, 1\}$.

3 Further analysis of zeta functional equations

Theorem 3.1. *Given that s is any (real of complex) number, except 0 and 1, then*

$$\begin{aligned} \log\left(\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\right) &= \log\left(\frac{\zeta\left(\frac{s+1}{2}\right)}{\zeta\left(1-\left(\frac{s+1}{2}\right)\right)} \frac{\zeta\left(\frac{s}{2}\right)}{\zeta\left(1-\frac{s}{2}\right)}\right) - \left(s - \frac{1}{2}\right) \log(2\pi) - \log\left(\frac{\sin\left(\frac{\pi\left(\frac{s+1}{2}\right)}{2}\right)}{\sin\left(\frac{\pi\left(1-\frac{s}{2}\right)}{2}\right)}\right); \\ \log\left(\frac{\zeta(s)}{\zeta(1-s)}\right) &= \log\left(\frac{\zeta\left(\frac{s+1}{2}\right)}{\zeta\left(1-\left(\frac{s+1}{2}\right)\right)} \frac{\zeta\left(\frac{s}{2}\right)}{\zeta\left(1-\frac{s}{2}\right)}\right) - \left(s - \frac{1}{2}\right) \log(2) - \log\left(\frac{\sin\left(\frac{\pi\left(\frac{s+1}{2}\right)}{2}\right)}{\sin\left(\frac{\pi\left(1-\frac{s}{2}\right)}{2}\right)}\right) \end{aligned} \quad (21)$$

Proof. Observe $\log(\frac{\zeta(s)}{\zeta(1-s)}) - s \log(2\pi) + \log(\pi) - \log(\sin(\frac{\pi s}{2})) = \log(\Gamma(1-s))$ which implies $\log(\frac{\zeta(1-s)}{\zeta(s)}) - (1-s) \log(2\pi) + \log(\pi) - \log(\cos(\frac{\pi s}{2})) = \log(\Gamma(s))$. Since $\frac{1-s}{2} = 1 - (\frac{s+1}{2})$, $\frac{s}{2} = 1 - (1 - \frac{s}{2})$, the following identities are valid:

$$\log(\Gamma(\frac{1-s}{2})) = \log(\Gamma(1 - (\frac{s+1}{2}))) = \log(\frac{\zeta(\frac{s+1}{2})}{\zeta(1 - (\frac{s+1}{2}))}) - (\frac{s+1}{2}) \log(2\pi) + \log(\pi) - \log(\sin(\frac{\pi(\frac{s+1}{2})}{2})); \quad (22)$$

$$\log(\Gamma(\frac{s}{2})) = \log(\Gamma(1 - (1 - \frac{s}{2}))) = \log(\frac{\zeta(1 - \frac{s}{2})}{\zeta(1 - (1 - \frac{s}{2}))}) - (1 - \frac{s}{2}) \log(2\pi) + \log(\pi) - \log(\sin(\frac{\pi(1 - \frac{s}{2})}{2})). \quad (23)$$

Subtracting eq. 23 from 22 gives:

$$\begin{aligned} \log(\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}) &= \log(\frac{\zeta(\frac{s+1}{2})}{\zeta(1 - (\frac{s+1}{2}))} \frac{\zeta(\frac{s}{2})}{\zeta(1 - \frac{s}{2})}) - (s - \frac{1}{2}) \log(2\pi) - \log(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1 - \frac{s}{2})}{2})}); \\ \log(\frac{\zeta(s)}{\zeta(1-s)}) &= \log(\frac{\zeta(\frac{s+1}{2})}{\zeta(1 - (\frac{s+1}{2}))} \frac{\zeta(\frac{s}{2})}{\zeta(1 - \frac{s}{2})}) - (s - \frac{1}{2}) \log(2) - \log(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1 - \frac{s}{2})}{2})}) \end{aligned}$$

since $\log(\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}) = \log(\frac{\zeta(s)}{\zeta(1-s)}) - (s - \frac{1}{2}) \log \pi$. □

The following identities demonstrate a combinatorial perspective on $\log(\frac{\zeta(s)}{\zeta(1-s)})$ decomposition:

Lemma 3.2.

$$\begin{aligned} \log(\frac{\zeta(s+1)}{\zeta(-s)}) &= \log(\frac{\zeta(\frac{s+1}{2})}{\zeta(1 - (\frac{s+1}{2}))} \frac{\zeta(\frac{s+2}{2})}{\zeta(1 - \frac{s+2}{2})}) - (s + \frac{1}{2}) \log(2) - \log(\frac{\sin(\frac{\pi(\frac{s+2}{2})}{2})}{\sin(\frac{\pi(1 - \frac{s+1}{2})}{2})}); \\ \log(\frac{\zeta(s)}{\zeta(1-s)}) &= \log(\frac{\zeta(\frac{s}{2})}{\zeta(1 - \frac{s}{2})} \frac{\zeta(\frac{s+1}{2})}{\zeta(1 - (\frac{s+1}{2}))}) - (s - \frac{1}{2}) \log(2) - \log(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1 - \frac{s}{2})}{2})}); \\ \log(\frac{\zeta(s-1)}{\zeta(2-s)}) &= \log(\frac{\zeta(\frac{s-1}{2})}{\zeta(1 - (\frac{s-1}{2}))} \frac{\zeta(\frac{s}{2})}{\zeta(1 - \frac{s}{2})}) - (s - \frac{3}{2}) \log(2) - \log(\frac{\sin(\frac{\pi(\frac{s}{2})}{2})}{\sin(\frac{\pi(1 - \frac{s-1}{2})}{2})}); \\ \log(\frac{\zeta(s-2)}{\zeta(3-s)}) &= \log(\frac{\zeta(\frac{s-2}{2})}{\zeta(1 - (\frac{s-2}{2}))} \frac{\zeta(\frac{s-1}{2})}{\zeta(1 - \frac{s-1}{2})}) - (s - \frac{5}{2}) \log(2) - \log(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1 - \frac{s-2}{2})}{2})}). \end{aligned} \quad (24)$$

Many important identities are derivable for this last set of equations.

Theorem 3.3.

$$\log(\frac{\zeta(s + \frac{1}{2})}{\zeta(\frac{1}{2} - s)} \frac{\zeta(s)}{\zeta(1-s)}) - \log(\frac{\sin(\frac{\pi}{2}(s + \frac{1}{2}))}{\sin(\frac{\pi}{2}(1-s))}) = \log(\frac{\zeta(s - \frac{1}{2})}{\zeta(\frac{3}{2} - s)} \frac{\zeta(s-1)}{\zeta(2-s)}) - \log(\frac{(s-1)(s - \frac{1}{2})}{(2\pi i)^2}) - \log(\frac{\sin(\frac{\pi}{2}(s - \frac{1}{2}))}{\sin(\frac{\pi}{2}(2-s))}) \quad (25)$$

Proof. Note

$$\log(\frac{\zeta(s-2)}{\zeta(3-s)}) = \log(\frac{\zeta(\frac{s-2}{2})}{\zeta(1 - \frac{s-2}{2})} \frac{\zeta(\frac{s-1}{2})}{\zeta(1 - (\frac{s-1}{2}))}) - (s - \frac{5}{2}) \log(2) - \log(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1 - \frac{s-2}{2})}{2})}),$$

and ¹.

$$\begin{aligned}
\log\left(\frac{(s-2)(s-1)}{(2\pi i)^2} \frac{\zeta(s)}{\zeta(1-s)}\right) &= \log\left(\frac{\zeta(\frac{s-2}{2})}{\zeta(1-\frac{s-2}{2})} \frac{\zeta(\frac{s-1}{2})}{\zeta(1-\frac{s-1}{2})}\right) - (s-\frac{5}{2})\log(2) - \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right) \\
\Rightarrow \log\left(\frac{\zeta(s)}{\zeta(1-s)}\right) &= \log\left(\frac{\zeta(\frac{s}{2})}{\zeta(1-\frac{s}{2})} \frac{\zeta(\frac{s+1}{2})}{\zeta(1-\frac{s+1}{2})}\right) - (s-\frac{1}{2})\log(2) - \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) \\
\log\left(\frac{\zeta(\frac{s}{2})}{\zeta(1-\frac{s}{2})} \frac{\zeta(\frac{s+1}{2})}{\zeta(1-\frac{s+1}{2})}\right) &= \log\left(\frac{\zeta(\frac{s-2}{2})}{\zeta(1-\frac{s-2}{2})} \frac{\zeta(\frac{s-1}{2})}{\zeta(1-\frac{s-1}{2})}\right) + \\
+ 2\log(2) - \log\left(\frac{(s-2)(s-1)}{(2\pi i)^2}\right) &+ \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right)
\end{aligned} \tag{26}$$

This suggests

$$\begin{aligned}
\log\left(\frac{\zeta(s+\frac{1}{2})}{\zeta(\frac{1}{2}-s)} \frac{\zeta(s)}{\zeta(1-s)}\right) &= \log\left(\frac{\zeta(s-\frac{1}{2})}{\zeta(\frac{3}{2}-s)} \frac{\zeta(s-1)}{\zeta(2-s)}\right) \\
- \log\left(\frac{(s-1)(s-\frac{1}{2})}{(2\pi i)^2}\right) &+ \log\left(\frac{\sin(\frac{\pi(s+\frac{1}{2})}{2})}{\sin(\frac{\pi}{2}(1-s))}\right) - \log\left(\frac{\sin(\frac{\pi(s-\frac{1}{2})}{2})}{\sin(\frac{\pi}{2}(2-s))}\right)
\end{aligned} \tag{27}$$

from which the identity

$$\log\left(\frac{\zeta(s+\frac{1}{2})}{\zeta(\frac{1}{2}-s)} \frac{\zeta(s)}{\zeta(1-s)}\right) - \log\left(\frac{\sin(\frac{\pi(s+\frac{1}{2})}{2})}{\sin(\frac{\pi}{2}(1-s))}\right) = \log\left(\frac{\zeta(s-\frac{1}{2})}{\zeta(\frac{3}{2}-s)} \frac{\zeta(s-1)}{\zeta(2-s)}\right) - \log\left(\frac{(s-1)(s-\frac{1}{2})}{(2\pi i)^2}\right) - \log\left(\frac{\sin(\frac{\pi(s-\frac{1}{2})}{2})}{\sin(\frac{\pi}{2}(2-s))}\right)$$

is derived. \square

3.1 Zeta at $1+s$, $-s$, $1-s$ and s

Theorem 3.4.

$$\begin{aligned}
\log\left(\frac{\zeta^2(1+s)}{\zeta^2(-s)}\right) &= -\log\left(\frac{\zeta^2(1-s)}{\zeta^2(s)}\right) - \log\left(\frac{s^2}{(2\pi i)^2}\right) + \\
- \log\left(\frac{\sin(\frac{\pi(\frac{s+2}{2})}{2})}{\sin(\frac{\pi(1-\frac{s+1}{2})}{2})}\right) &+ \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right)
\end{aligned} \tag{28}$$

Proof. According to lemma 3.2,

$$\begin{aligned}
\log\left(\frac{\zeta(s+1)}{\zeta(-s)}\right) - \log\left(\frac{\zeta(s)}{\zeta(1-s)}\right) &+ \log\left(\frac{\zeta(s-1)}{\zeta(2-s)}\right) - \log\left(\frac{\zeta(s-2)}{\zeta(3-s)}\right) - \left(\log\left(\frac{\zeta(\frac{s+2}{2})}{\zeta(1-\frac{s+2}{2})}\right) - \log\left(\frac{\zeta(\frac{s-2}{2})}{\zeta(1-\frac{s-2}{2})}\right)\right) \\
= -2\log 2 - \log\left(\frac{\sin(\frac{\pi(\frac{s+2}{2})}{2})}{\sin(\frac{\pi(1-\frac{s+1}{2})}{2})}\right) &+ \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right)
\end{aligned} \tag{29}$$

¹ $\frac{\zeta(s-2)}{\zeta(3-s)} = \frac{(s-2)(s-1)}{(2\pi i)^2} \frac{\zeta(s)}{\zeta(1-s)}$ - an independent discovery by the author similar in comparison to one of Henrik Stenlund's results[5].

The recursion relations between the first six terms may be combined to produce:

$$\begin{aligned}
& \log\left(\frac{(\mathbf{s}-\mathbf{1})\mathbf{s}}{(2\pi i)^2} \frac{\zeta^2(s+1)}{\zeta^2(-s)}\right) - \log\left(\frac{(\mathbf{s}-\mathbf{2})(\mathbf{s}-\mathbf{1})}{(2\pi i)^2} \frac{\zeta^2(s)}{\zeta^2(1-s)}\right) - \left(\log\left(\frac{\zeta(\frac{s+2}{2})}{\zeta(1-\frac{s+2}{2})}\right) - \log\left(\frac{(\frac{\mathbf{s}}{2}-\mathbf{1})\frac{\mathbf{s}}{2}}{(2\pi i)^2} \frac{\zeta(\frac{s+2}{2})}{\zeta(1-\frac{s+2}{2})}\right) \right) \\
&= -2\log 2 - \log\left(\frac{\sin(\frac{\pi(\frac{s+2}{2})}{2})}{\sin(\frac{\pi(1-\frac{s+1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{\mathbf{s}}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{\mathbf{s}}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right)
\end{aligned} \tag{30}$$

or alternatively,

$$\begin{aligned}
& \log\left(\frac{(\mathbf{s}-\mathbf{1})\mathbf{s}}{(2\pi i)^2} \frac{\zeta^2(s+1)}{\zeta^2(-s)}\right) - \log\left(\frac{(\mathbf{s}-\mathbf{2})(\mathbf{s}-\mathbf{1})}{(2\pi i)^2} \frac{\zeta^2(s)}{\zeta^2(1-s)}\right) - \left(\log\left(\frac{(2\pi i)^2}{-\frac{\mathbf{s}}{2}(1-\frac{\mathbf{s}}{2})} \frac{\zeta(\frac{s-2}{2})}{\zeta(1-\frac{s-2}{2})}\right) - \log\left(\frac{\zeta(\frac{s-2}{2})}{\zeta(1-\frac{s-2}{2})}\right) \right) \\
&= -2\log 2 - \log\left(\frac{\sin(\frac{\pi(\frac{s+2}{2})}{2})}{\sin(\frac{\pi(1-\frac{s+1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{\mathbf{s}}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{\mathbf{s}}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right)
\end{aligned} \tag{31}$$

As a result of this aggregation

$$\begin{aligned}
& \log\left(\frac{(\mathbf{s}-\mathbf{1})\mathbf{s}}{(2\pi i)^2} \frac{\zeta^2(s+1)}{\zeta^2(-s)}\right) - \log\left(\frac{(\mathbf{s}-\mathbf{2})(\mathbf{s}-\mathbf{1})}{(2\pi i)^2} \frac{\zeta^2(s)}{\zeta^2(1-s)}\right) + \log\left(\frac{(\frac{\mathbf{s}}{2}-\mathbf{1})\frac{\mathbf{s}}{2}}{(2\pi i)^2}\right) \\
&= -2\log 2 - \log\left(\frac{\sin(\frac{\pi(\frac{s+2}{2})}{2})}{\sin(\frac{\pi(1-\frac{s+1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{\mathbf{s}}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{\mathbf{s}}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right)
\end{aligned} \tag{32}$$

is produced. Further aggregation results in

$$\begin{aligned}
& \log\left(\frac{\mathbf{s}}{\mathbf{s}-\mathbf{2}} \frac{\zeta^2(1+s)}{\zeta^2(-s)} \frac{\zeta^2(1-s)}{\zeta^2(s)}\right) = -\log\left(\frac{(\frac{\mathbf{s}}{2}-\mathbf{1})\frac{\mathbf{s}}{2}}{(2\pi i)^2}\right) - 2\log 2 \\
& - \log\left(\frac{\sin(\frac{\pi(\frac{s+2}{2})}{2})}{\sin(\frac{\pi(1-\frac{s+1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{\mathbf{s}}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{\mathbf{s}}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right) \\
& \Rightarrow
\end{aligned} \tag{33}$$

$$\begin{aligned}
& \log\left(\frac{\zeta^2(1+s)}{\zeta^2(-s)} \frac{\zeta^2(1-s)}{\zeta^2(s)}\right) = -\log\left(\frac{\mathbf{s}}{\mathbf{s}-\mathbf{2}}\right) - \log\left(\frac{(\frac{\mathbf{s}}{2}-\mathbf{1})\frac{\mathbf{s}}{2}}{(2\pi i)^2}\right) - 2\log 2 \\
& - \log\left(\frac{\sin(\frac{\pi(\frac{s+2}{2})}{2})}{\sin(\frac{\pi(1-\frac{s+1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{\mathbf{s}}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{\mathbf{s}}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right) \\
& \Rightarrow
\end{aligned} \tag{34}$$

$$\begin{aligned}
& \log\left(\frac{\zeta^2(1+s)}{\zeta^2(-s)} \frac{\zeta^2(1-s)}{\zeta^2(s)}\right) = -\log\left(\frac{2\mathbf{s}^2}{(\mathbf{s}-\mathbf{2})} \frac{(\frac{\mathbf{s}}{2}-\mathbf{1})}{(2\pi i)^2}\right) + \\
& - \log\left(\frac{\sin(\frac{\pi(\frac{s+2}{2})}{2})}{\sin(\frac{\pi(1-\frac{s+1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{\mathbf{s}}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{\mathbf{s}}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right)
\end{aligned} \tag{35}$$

and finally

$$\begin{aligned} \log\left(\frac{\zeta^2(1+s)}{\zeta^2(-s)}\right) &= -\log\left(\frac{\zeta^2(1-s)}{\zeta^2(s)}\right) - \log\left(\frac{s^2}{(2\pi i)^2}\right) + \\ &- \log\left(\frac{\sin(\frac{\pi(\frac{s+2}{2})}{2})}{\sin(\frac{\pi(1-\frac{s+1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right). \end{aligned}$$

□

3.2 On the decomposition of the derivatives of logarithm of zeta

Here we would like to prove the following

Theorem 3.5.

$$\begin{aligned} &\log\left(\frac{(s-2)(s-1)}{(2\pi i)^2} \frac{\zeta^2(s)}{\zeta^2(1-s)}\right) \\ &= \log\left(\frac{\zeta(\frac{s+2}{2})}{\zeta(1-\frac{s+2}{2})} \frac{\zeta(\frac{s+1}{2})}{\zeta(1-\frac{s+1}{2})} \frac{\zeta(\frac{s}{2})}{\zeta(1-\frac{s}{2})} \frac{\zeta(\frac{s-1}{2})}{\zeta(1-\frac{s-1}{2})}\right) + \log\left(\frac{(\frac{s}{2}-1)\frac{s}{2}}{(2\pi i)^2}\right) - (2s-3)\log 2 + \\ &\quad - \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right) \\ &= \log\left(\frac{\zeta(\frac{s+1}{2})}{\zeta(1-\frac{s+1}{2})} \frac{\zeta(\frac{s}{2})}{\zeta(1-\frac{s}{2})} \frac{\zeta(\frac{s-1}{2})}{\zeta(1-\frac{s-1}{2})} \frac{\zeta(\frac{s-2}{2})}{\zeta(1-\frac{s-2}{2})}\right) - (2s-3)\log 2 - \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right) \end{aligned} \quad (36)$$

Proof. Again, according to lemma 3.2,

$$\begin{aligned} &\log\left(\frac{\zeta(s+1)}{\zeta(-s)}\right) + \log\left(\frac{\zeta(s)}{\zeta(1-s)}\right) + \log\left(\frac{\zeta(s-1)}{\zeta(2-s)}\right) + \log\left(\frac{\zeta(s-2)}{\zeta(3-s)}\right) - \left(\log\left(\frac{\zeta(\frac{s+2}{2})}{\zeta(1-\frac{s+2}{2})}\right) + \log\left(\frac{\zeta(\frac{s-2}{2})}{\zeta(1-\frac{s-2}{2})}\right)\right) \\ &\quad - \left(\log\left(\frac{\zeta^2(\frac{s+1}{2})}{\zeta^2(1-\frac{s+1}{2})}\right) + \log\left(\frac{\zeta^2(\frac{s}{2})}{\zeta^2(1-\frac{s}{2})}\right) + \log\left(\frac{\zeta^2(\frac{s-1}{2})}{\zeta^2(1-\frac{s-1}{2})}\right)\right) \\ &= -(4s-4)\log 2 - \log\left(\frac{\sin(\frac{\pi(\frac{s+2}{2})}{2})}{\sin(\frac{\pi(1-\frac{s+1}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-1}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right) \end{aligned} \quad (37)$$

\Rightarrow

$$\begin{aligned} &\log\left(\frac{(s-1)s}{(2\pi i)^2} \frac{\zeta^2(s+1)}{\zeta^2(-s)}\right) + \log\left(\frac{(s-2)(s-1)}{(2\pi i)^2} \frac{\zeta^2(s)}{\zeta^2(1-s)}\right) - \left(\log\left(\frac{\zeta(\frac{s+2}{2})}{\zeta(1-\frac{s+2}{2})}\right) + \log\left(\frac{\zeta(\frac{s-2}{2})}{\zeta(1-\frac{s-2}{2})}\right)\right) \\ &\quad - \left(\log\left(\frac{\zeta^2(\frac{s+1}{2})}{\zeta^2(1-\frac{s+1}{2})}\right) + \log\left(\frac{\zeta^2(\frac{s}{2})}{\zeta^2(1-\frac{s}{2})}\right) + \log\left(\frac{\zeta^2(\frac{s-1}{2})}{\zeta^2(1-\frac{s-1}{2})}\right)\right) \\ &= -(4s-4)\log 2 - \log\left(\frac{\sin(\frac{\pi(\frac{s+2}{2})}{2})}{\sin(\frac{\pi(1-\frac{s+1}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-1}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right). \end{aligned} \quad (38)$$

From this last equation we subtract eq. 32:

$$\begin{aligned} & \log\left(\frac{(\mathbf{s}-\mathbf{1})\mathbf{s}}{(2\pi i)^2} \frac{\zeta^2(s+1)}{\zeta^2(-s)}\right) - \log\left(\frac{(\mathbf{s}-\mathbf{2})(\mathbf{s}-\mathbf{1})}{(2\pi i)^2} \frac{\zeta^2(s)}{\zeta^2(1-s)}\right) + \log\left(\frac{(\frac{\mathbf{s}}{2}-\mathbf{1})\frac{\mathbf{s}}{2}}{(2\pi i)^2}\right) \\ &= -2\log 2 - \log\left(\frac{\sin(\frac{\pi(\frac{s+2}{2})}{2})}{\sin(\frac{\pi(1-\frac{s+1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right) \end{aligned}$$

to yield

$$\begin{aligned} & 2\log\left(\frac{(\mathbf{s}-\mathbf{2})(\mathbf{s}-\mathbf{1})}{(2\pi i)^2} \frac{\zeta^2(s)}{\zeta^2(1-s)}\right) = \\ & \left(\log\left(\frac{\zeta(\frac{s+2}{2})}{\zeta(1-\frac{s+2}{2})}\right) + \log\left(\frac{\zeta(\frac{s-2}{2})}{\zeta(1-\frac{s-2}{2})}\right)\right) + \left(\log\left(\frac{\zeta^2(\frac{s+1}{2})}{\zeta^2(1-\frac{s+1}{2})}\right) + \log\left(\frac{\zeta^2(\frac{s}{2})}{\zeta^2(1-\frac{s}{2})}\right) + \log\left(\frac{\zeta^2(\frac{s-1}{2})}{\zeta^2(1-\frac{s-1}{2})}\right)\right) \\ & + \log\left(\frac{(\frac{\mathbf{s}}{2}-\mathbf{1})\frac{\mathbf{s}}{2}}{(2\pi i)^2}\right) - (4\mathbf{s}-6)\log 2 - 2\log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) - 2\log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right) \end{aligned} \quad (39)$$

which shows the decomposition relation of $\log(\frac{\zeta^2(s)}{\zeta^2(1-s)})$ as

$$\begin{aligned} & 2\log\left(\frac{(\mathbf{s}-\mathbf{2})(\mathbf{s}-\mathbf{1})}{(2\pi i)^2} \frac{\zeta^2(s)}{\zeta^2(1-s)}\right) = \\ & \left(\log\left(\frac{(\frac{\mathbf{s}}{2}-\mathbf{1})\frac{\mathbf{s}}{2}}{(2\pi i)^2} \frac{\zeta^2(\frac{s+2}{2})}{\zeta^2(1-\frac{s+2}{2})}\right)\right) + \left(\log\left(\frac{\zeta^2(\frac{s+1}{2})}{\zeta^2(1-\frac{s+1}{2})}\right) + \log\left(\frac{\zeta^2(\frac{s}{2})}{\zeta^2(1-\frac{s}{2})}\right) + \log\left(\frac{\zeta^2(\frac{s-1}{2})}{\zeta^2(1-\frac{s-1}{2})}\right)\right) \\ & + \log\left(\frac{(\frac{\mathbf{s}}{2}-\mathbf{1})\frac{\mathbf{s}}{2}}{(2\pi i)^2}\right) - (4\mathbf{s}-6)\log 2 - 2\log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) - 2\log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right) \end{aligned} \quad (40)$$

simplified to

$$\begin{aligned} \log\left(\frac{(\mathbf{s}-\mathbf{2})(\mathbf{s}-\mathbf{1})}{(2\pi i)^2} \frac{\zeta^2(s)}{\zeta^2(1-s)}\right) &= \log\left(\frac{\zeta(\frac{s+2}{2})}{\zeta(1-\frac{s+2}{2})} \frac{\zeta(\frac{s+1}{2})}{\zeta(1-\frac{s+1}{2})} \frac{\zeta(\frac{s}{2})}{\zeta(1-\frac{s}{2})} \frac{\zeta(\frac{s-1}{2})}{\zeta(1-\frac{s-1}{2})}\right) \\ &+ \log\left(\frac{(\frac{\mathbf{s}}{2}-\mathbf{1})\frac{\mathbf{s}}{2}}{(2\pi i)^2}\right) - (2\mathbf{s}-3)\log 2 - \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right) \end{aligned} \quad (41)$$

or

$$\begin{aligned} \log\left(\frac{(\mathbf{s}-\mathbf{2})(\mathbf{s}-\mathbf{1})}{(2\pi i)^2} \frac{\zeta^2(s)}{\zeta^2(1-s)}\right) &= \log\left(\frac{\zeta(\frac{s+1}{2})}{\zeta(1-\frac{s+1}{2})} \frac{\zeta(\frac{s}{2})}{\zeta(1-\frac{s}{2})} \frac{\zeta(\frac{s-1}{2})}{\zeta(1-\frac{s-1}{2})} \frac{\zeta(\frac{s-2}{2})}{\zeta(1-\frac{s-2}{2})}\right) \\ &- (2\mathbf{s}-3)\log 2 - \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right) \end{aligned} \quad (42)$$

depending on the identity

$$\frac{\zeta(\frac{s-2}{2})}{\zeta(1-\frac{s-2}{2})} = \log\left(\frac{(\frac{\mathbf{s}}{2}-\mathbf{1})\frac{\mathbf{s}}{2}}{(2\pi i)^2}\right) + \log\left(\frac{\zeta(\frac{s+2}{2})}{\zeta(1-\frac{s+2}{2})}\right). \quad (43)$$

□

Lemma 3.6. *The following identity is true.*

$$\log\left(\frac{\sin(\frac{\pi}{2}(\frac{s}{2} + \frac{1}{2}))}{\cos(\frac{\pi}{2}(\frac{s}{2}))}\right) = \log\left(\frac{\sin(\frac{\pi}{2}(\frac{s-4}{2} + \frac{1}{2}))}{\cos(\frac{\pi}{2}(\frac{s-4}{2}))}\right). \quad (44)$$

Proof.

$$\begin{aligned} \log\left(\frac{\zeta^2(s)}{\zeta^2(1-s)}\right) &= \log\left(\frac{\zeta(\frac{s+1}{2})}{\zeta(1-\frac{s+1}{2})} \frac{\zeta(\frac{s}{2})}{\zeta(1-\frac{s}{2})} \frac{\zeta(\frac{s-1}{2})}{\zeta(1-\frac{s-1}{2})} \frac{\zeta(\frac{s-2}{2})}{\zeta(1-\frac{s-2}{2})}\right) \\ &\quad - \log\left(\frac{(s-2)(s-1)}{(2\pi i)^2}\right) - (2s-3)\log 2 - \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right) \end{aligned} \quad (45)$$

Recall the identity

$$\log\left(\frac{\zeta(s)}{\zeta(1-s)}\right) = \log\left(\frac{\zeta(\frac{s}{2})}{\zeta(1-\frac{s}{2})} \frac{\zeta(\frac{s+1}{2})}{\zeta(1-\frac{s+1}{2})}\right) - (s-\frac{1}{2})\log(2) - \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right),$$

therefore

$$\begin{aligned} \log\left(\frac{\zeta^2(\frac{s}{2})}{\zeta^2(1-\frac{s}{2})} \frac{\zeta^2(\frac{s+1}{2})}{\zeta^2(1-\frac{s+1}{2})}\right) &- (2s-1)\log(2) - 2\log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) = \\ &\log\left(\frac{\zeta(\frac{s+1}{2})}{\zeta(1-\frac{s+1}{2})} \frac{\zeta(\frac{s}{2})}{\zeta(1-\frac{s}{2})} \frac{\zeta(\frac{s-1}{2})}{\zeta(1-\frac{s-1}{2})} \frac{\zeta(\frac{s-2}{2})}{\zeta(1-\frac{s-2}{2})}\right) \\ &- \log\left(\frac{(s-2)(s-1)}{(2\pi i)^2}\right) - (2s-3)\log 2 - \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right) \end{aligned} \quad (46)$$

implying

$$\begin{aligned} &\log\left(\frac{\zeta(\frac{s}{2})}{\zeta(1-\frac{s}{2})} \frac{\zeta(\frac{s+1}{2})}{\zeta(1-\frac{s+1}{2})}\right) - (2s-1)\log(2) - 2\log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) = \\ &\log\left(\frac{\zeta(\frac{s-1}{2})}{\zeta(1-\frac{s-1}{2})} \frac{\zeta(\frac{s-2}{2})}{\zeta(1-\frac{s-2}{2})}\right) - \log\left(\frac{(s-2)(s-1)}{(2\pi i)^2}\right) - (2s-3)\log 2 - \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right) \end{aligned} \quad (47)$$

$$\begin{aligned} \log\left(\frac{\zeta(\frac{s+1}{2})}{\zeta(1-\frac{s+1}{2})} \frac{\zeta(\frac{s}{2})}{\zeta(1-\frac{s}{2})}\right) &= \log\left(\frac{\zeta(\frac{s-1}{2})}{\zeta(1-\frac{s-1}{2})} \frac{\zeta(\frac{s-2}{2})}{\zeta(1-\frac{s-2}{2})}\right) + (2)\log 2 - \log\left(\frac{(s-2)(s-1)}{(2\pi i)^2}\right) + \\ &\quad + \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right) \end{aligned} \quad (48)$$

Changing s such that $s \rightarrow s-2$ leads to:

$$\begin{aligned} \log\left(\frac{\zeta(\frac{s-1}{2})}{\zeta(1-\frac{s-1}{2})} \frac{\zeta(\frac{s-2}{2})}{\zeta(1-\frac{s-2}{2})}\right) &= \log\left(\frac{\zeta(\frac{s-3}{2})}{\zeta(1-\frac{s-3}{2})} \frac{\zeta(\frac{s-4}{2})}{\zeta(1-\frac{s-4}{2})}\right) + \\ &\quad + (2)\log 2 - \log\left(\frac{(s-4)(s-3)}{(2\pi i)^2}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s-3}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-4}{2})}{2})}\right). \end{aligned} \quad (49)$$

Substituting this last result into eq. 48 produces

$$\begin{aligned}
& \log\left(\frac{\zeta(\frac{s+1}{2})}{\zeta(1-\frac{s+1}{2})} \frac{\zeta(\frac{s}{2})}{\zeta(1-\frac{s}{2})}\right) = \left(\log\left(\frac{\zeta(\frac{s-3}{2})}{\zeta(1-\frac{s-3}{2})} \frac{\zeta(\frac{s-4}{2})}{\zeta(1-\frac{s-4}{2})}\right) + \right. \\
& + (2) \log 2 - \log\left(\frac{(s-4)(s-3)}{(2\pi i)^2}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})} - \log\left(\frac{\sin(\frac{\pi(\frac{s-3}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-4}{2})}{2})}\right) \right) \\
& + (2) \log 2 - \log\left(\frac{(s-2)(s-1)}{(2\pi i)^2}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})} - \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right) \right)
\end{aligned} \tag{50}$$

and

$$\begin{aligned}
& \log\left(\frac{\zeta(\frac{s+1}{2})}{\zeta(1-\frac{s+1}{2})} \frac{\zeta(\frac{s}{2})}{\zeta(1-\frac{s}{2})}\right) = \log\left(\frac{\zeta(\frac{s-3}{2})}{\zeta(1-\frac{s-3}{2})} \frac{\zeta(\frac{s-4}{2})}{\zeta(1-\frac{s-4}{2})}\right) + \\
& + (4) \log 2 - \log\left(\frac{(s-1)(s-2)(s-3)(s-4)}{(2\pi i)^4}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})} - \log\left(\frac{\sin(\frac{\pi(\frac{s-3}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-4}{2})}{2})}\right).
\end{aligned} \tag{51}$$

Taking an analytical look at the right hand term

$$\log\left(\frac{\zeta(\frac{s-3}{2})}{\zeta(1-\frac{s-3}{2})} \frac{\zeta(\frac{s-4}{2})}{\zeta(1-\frac{s-4}{2})}\right) = \log\left(\frac{\frac{s-3}{2} \frac{s-1}{2}}{(2\pi i)^2} \frac{\zeta(\frac{s+1}{2})}{\zeta(1-\frac{s+1}{2})} \frac{\frac{s-4}{2} \frac{s-2}{2}}{(2\pi i)^2} \frac{\zeta(\frac{s}{2})}{\zeta(1-\frac{s}{2})}\right) \tag{52}$$

implies

$$\begin{aligned}
& \log\left(\frac{\zeta(\frac{s+1}{2})}{\zeta(1-\frac{s+1}{2})} \frac{\zeta(\frac{s}{2})}{\zeta(1-\frac{s}{2})}\right) = \log\left(\frac{\frac{s-3}{2} \frac{s-1}{2}}{(2\pi i)^2} \frac{\zeta(\frac{s+1}{2})}{\zeta(1-\frac{s+1}{2})} \frac{\frac{s-4}{2} \frac{s-2}{2}}{(2\pi i)^2} \frac{\zeta(\frac{s}{2})}{\zeta(1-\frac{s}{2})}\right) + \\
& + (4) \log 2 - \log\left(\frac{(s-1)(s-2)(s-3)(s-4)}{(2\pi i)^4}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})} - \log\left(\frac{\sin(\frac{\pi(\frac{s-3}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-4}{2})}{2})}\right)
\end{aligned} \tag{53}$$

and as a result of further simplification,

$$\log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) = \log\left(\frac{\sin(\frac{\pi(\frac{s+1-4}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-4}{2})}{2})}\right); \tag{54}$$

confirming

$$\log\left(\frac{\sin(\frac{\pi}{2}(\frac{s}{2} + \frac{1}{2}))}{\cos(\frac{\pi}{2}(\frac{s}{2}))}\right) = \log\left(\frac{\sin(\frac{\pi}{2}(\frac{s-4}{2} + \frac{1}{2}))}{\cos(\frac{\pi}{2}(\frac{s-4}{2}))}\right). \tag{55}$$

□

4 Further analysis involving the gamma function

Here, we first prove the following theorem before presenting the related identity in terms of zeta function.

Theorem 4.1.

$$\log\left(\frac{\Gamma(2+\frac{s}{2})}{\Gamma(\frac{1}{2}-2-\frac{s}{2})}\right) - \log\left(\frac{\Gamma(2-\frac{s}{2})}{\Gamma(\frac{1}{2}-2+\frac{s}{2})}\right) = \log\left(\frac{\Gamma(\frac{s}{2}-2)}{\Gamma(\frac{1}{2}+2-\frac{s}{2})}\right) - \log\left(\frac{\Gamma(-\frac{s}{2}-2)}{\Gamma(\frac{1}{2}+2+\frac{s}{2})}\right) + \log\left(\frac{s-4}{s+4}\right) \quad (56)$$

Proof. Due to theorem 3.4:

$$\begin{aligned} \log\left(\frac{\Gamma^2(-\frac{s}{2})}{\Gamma^2(\frac{1+s}{2})}\right) + \log(\pi^{2(1+s-\frac{1}{2})}) &= -\log\left(\frac{\Gamma^2(\frac{s}{2})}{\Gamma^2(\frac{1-s}{2})}\right) - \log(\pi^{2(1-s-\frac{1}{2})}) - \log\left(\frac{2s^2}{(s-2)} \frac{(\frac{s}{2}-1)}{(2\pi i)^2}\right) + \\ &\quad - \log\left(\frac{\sin(\frac{\pi(\frac{s+2}{2})}{2})}{\sin(\frac{\pi(1-\frac{s+1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right). \end{aligned} \quad (57)$$

Also

$$\begin{aligned} &\log\left(\frac{\Gamma^2(-\frac{(s-4)}{2})}{\Gamma^2(\frac{1+s-4}{2})}\right) + \log(\pi^{2(1+s-4-\frac{1}{2})}) \\ &= -\log\left(\frac{\Gamma^2(\frac{s-4}{2})}{\Gamma^2(\frac{1-(s-4)}{2})}\right) - \log(\pi^{2(1-(s-4)-\frac{1}{2})}) - \log\left(\frac{2(s-4)^2}{(s-4-2)} \frac{(\frac{s-4}{2}-1)}{(2\pi i)^2}\right) + \\ &\quad - \log\left(\frac{\sin(\frac{\pi(\frac{s+2}{2})}{2})}{\sin(\frac{\pi(1-\frac{s+1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right), \end{aligned} \quad (58)$$

because of the following identities which are derived in relation to the identity 54:

$$\begin{aligned} \log\left(\frac{\sin(\frac{\pi(\frac{s+2}{2})}{2})}{\sin(\frac{\pi(1-\frac{s+1}{2})}{2})}\right) &= \log\left(\frac{\sin(\frac{\pi(\frac{s-4+2}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-4+1}{2})}{2})}\right); \\ \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) &= \log\left(\frac{\sin(\frac{\pi(\frac{s-4+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-4}{2})}{2})}\right); \\ &\vdots \end{aligned}$$

and so on. Subtracting eq. 58 from 57:

$$\begin{aligned} &\log\left(\frac{\Gamma^2(-\frac{s}{2})}{\Gamma^2(\frac{1+s}{2})}\right) + \log(\pi^{2(1+s-\frac{1}{2})}) - \left(\log\left(\frac{\Gamma^2(-\frac{(s-4)}{2})}{\Gamma^2(\frac{1+s-4}{2})}\right) + \log(\pi^{2(1+s-4-\frac{1}{2})})\right) = \\ &\quad - \log\left(\frac{\Gamma^2(\frac{s}{2})}{\Gamma^2(\frac{1-s}{2})}\right) - \log(\pi^{2(1-s-\frac{1}{2})}) - \log\left(\frac{2s^2}{(s-2)} \frac{(\frac{s}{2}-1)}{(2\pi i)^2}\right) \\ &\quad - \left(-\log\left(\frac{\Gamma^2(\frac{s-4}{2})}{\Gamma^2(\frac{1-(s-4)}{2})}\right) - \log(\pi^{2(1-(s-4)-\frac{1}{2})}) - \log\left(\frac{2(s-4)^2}{(s-4-2)} \frac{(\frac{s-4}{2}-1)}{(2\pi i)^2}\right)\right) \end{aligned} \quad (59)$$

\Rightarrow

$$\begin{aligned} \log\left(\frac{\Gamma^2(-\frac{s}{2})}{\Gamma^2(\frac{1+s}{2})}\right) - \log\left(\frac{\Gamma^2(2-\frac{s}{2})}{\Gamma^2(\frac{1+s}{2}-2)}\right) &= -\log\left(\frac{\Gamma^2(\frac{s}{2})}{\Gamma^2(\frac{1-s}{2})}\right) - \log\left(\frac{s^2}{(s-2)} \frac{(\frac{s}{2}-1)}{(2\pi i)^2}\right) + \\ &\quad + \log\left(\frac{\Gamma^2(\frac{s}{2}-2)}{\Gamma^2(2+\frac{1-s}{2})}\right) + \log\left(\frac{(s-4)^2}{(s-6)} \frac{(\frac{s}{2}-3)}{(2\pi i)^2}\right) \end{aligned} \quad (60)$$

\Rightarrow

$$\begin{aligned} & \log\left(\frac{\Gamma^2(-\frac{s}{2})}{\Gamma^2(\frac{1+s}{2})}\right) + \log\left(\frac{\Gamma^2(\frac{s}{2})}{\Gamma^2(\frac{1-s}{2})}\right) + \log\left(\frac{s^2}{(s-2)}\left(\frac{s}{2} - 1\right)\right) \\ &= \log\left(\frac{\Gamma^2(2 - \frac{s}{2})}{\Gamma^2(\frac{1+s}{2} - 2)}\right) + \log\left(\frac{\Gamma^2(\frac{s}{2} - 2)}{\Gamma^2(2 + \frac{1-s}{2})}\right) + \log\left(\frac{(s-4)^2}{(s-6)}\left(\frac{s}{2} - 3\right)\right) \end{aligned} \quad (61)$$

\Rightarrow

$$\log\left(\frac{\Gamma^2(-\frac{s}{2})}{\Gamma^2(\frac{1+s}{2})}\right) + \log\left(\frac{\Gamma^2(\frac{s}{2})}{\Gamma^2(\frac{1-s}{2})}\right) + \log\left(\frac{s^2}{2}\right) = \log\left(\frac{\Gamma^2(2 - \frac{s}{2})}{\Gamma^2(\frac{1+s}{2} - 2)}\right) + \log\left(\frac{\Gamma^2(\frac{s}{2} - 2)}{\Gamma^2(2 + \frac{1-s}{2})}\right) + \log\left(\frac{(s-4)^2}{2}\right) \quad (62)$$

\Rightarrow

$$\log\left(\frac{\Gamma^2(-\frac{s}{2})}{\Gamma^2(\frac{1+s}{2})}\right) + \log\left(\frac{\Gamma^2(\frac{s}{2})}{\Gamma^2(\frac{1-s}{2})}\right) + \log(s^2) = \log\left(\frac{\Gamma^2(2 - \frac{s}{2})}{\Gamma^2(\frac{1+s}{2} - 2)}\right) + \log\left(\frac{\Gamma^2(\frac{s}{2} - 2)}{\Gamma^2(2 + \frac{1-s}{2})}\right) + \log((s-4)^2) \quad (63)$$

\Rightarrow

$$\log\left(\frac{\Gamma(-\frac{s}{2})}{\Gamma(\frac{1}{2} + \frac{s}{2})}\right) + \log\left(\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1}{2} - \frac{s}{2})}\right) + \log(s) = \log\left(\frac{\Gamma(2 - \frac{s}{2})}{\Gamma(\frac{1}{2} - 2 + \frac{s}{2})}\right) + \log\left(\frac{\Gamma(\frac{s}{2} - 2)}{\Gamma(\frac{1}{2} + 2 - \frac{s}{2})}\right) + \log(s-4) \quad (64)$$

$s \rightarrow -s$:

$$\log\left(\frac{\Gamma(-\frac{s}{2})}{\Gamma(\frac{1}{2} + \frac{s}{2})}\right) + \log\left(\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1}{2} - \frac{s}{2})}\right) + \log(-s) = \log\left(\frac{\Gamma(2 + \frac{s}{2})}{\Gamma(\frac{1}{2} - 2 - \frac{s}{2})}\right) + \log\left(\frac{\Gamma(-\frac{s}{2} - 2)}{\Gamma(\frac{1}{2} + 2 + \frac{s}{2})}\right) + \log(-s-4). \quad (65)$$

Subtracting eq. 65 from 64:

$$\log\left(\frac{\Gamma(2 + \frac{s}{2})}{\Gamma(\frac{1}{2} - 2 - \frac{s}{2})}\right) + \log\left(\frac{\Gamma(-\frac{s}{2} - 2)}{\Gamma(\frac{1}{2} + 2 + \frac{s}{2})}\right) = \log\left(\frac{\Gamma(2 - \frac{s}{2})}{\Gamma(\frac{1}{2} - 2 + \frac{s}{2})}\right) + \log\left(\frac{\Gamma(\frac{s}{2} - 2)}{\Gamma(\frac{1}{2} + 2 - \frac{s}{2})}\right) + \log\left(\frac{s-4}{s+4}\right) \quad (66)$$

Therefore

$$\log\left(\frac{\Gamma(2 + \frac{s}{2})}{\Gamma(\frac{1}{2} - 2 - \frac{s}{2})}\right) - \log\left(\frac{\Gamma(2 - \frac{s}{2})}{\Gamma(\frac{1}{2} - 2 + \frac{s}{2})}\right) = \log\left(\frac{\Gamma(\frac{s}{2} - 2)}{\Gamma(\frac{1}{2} + 2 - \frac{s}{2})}\right) - \log\left(\frac{\Gamma(-\frac{s}{2} - 2)}{\Gamma(\frac{1}{2} + 2 + \frac{s}{2})}\right) + \log\left(\frac{s-4}{s+4}\right)$$

or

$$-\log\left(\frac{\Gamma(2 + \frac{s}{2})}{\Gamma(\frac{1}{2} - 2 - \frac{s}{2})}\right) + \log\left(\frac{\Gamma(2 - \frac{s}{2})}{\Gamma(\frac{1}{2} - 2 + \frac{s}{2})}\right) = -\log\left(\frac{\Gamma(\frac{s}{2} - 2)}{\Gamma(\frac{1}{2} + 2 - \frac{s}{2})}\right) + \log\left(\frac{\Gamma(-\frac{s}{2} - 2)}{\Gamma(\frac{1}{2} + 2 + \frac{s}{2})}\right) + \log\left(\frac{s+4}{s-4}\right)$$

as a result of $s \rightarrow -s$ substitution. \square

4.1 Logarithm of zeta at $\pm s \pm 4$ and $1 - (\pm s \pm 4)$

The related zeta version of the last equation may be presented in form of

$$\begin{aligned} & -\log\left(\frac{\zeta(s+4)}{\zeta(-3-s)}\pi^{\frac{1}{2}-(s+4)}\right) + \log\left(\frac{\zeta(-s+4)}{\zeta(-3+s)}\pi^{\frac{1}{2}-(-s+4)}\right) \\ &= -\log\left(\frac{\zeta(s-4)}{\zeta(5-s)}\pi^{\frac{1}{2}-(s-4)}\right) + \log\left(\frac{\zeta(-s-4)}{\zeta(5+s)}\pi^{\frac{1}{2}-(-s-4)}\right) + \log\left(\frac{s-4}{s+4}\right) \end{aligned} \quad (67)$$

\Rightarrow

$$\begin{aligned}
& -\log\left(\frac{\zeta(s+4)}{\zeta(-3-s)}\right) + \log\left(\frac{\zeta(-s+4)}{\zeta(-3+s)}\right) + \log\left(\frac{\pi^{\frac{1}{2}-(-s+4)}}{\pi^{\frac{1}{2}-(s+4)}}\right) \\
& = -\log\left(\frac{\zeta(s-4)}{\zeta(5-s)}\right) + \log\left(\frac{\zeta(-s-4)}{\zeta(5+s)}\right) + \log\left(\frac{\pi^{\frac{1}{2}-(-s-4)}}{\pi^{\frac{1}{2}-(s-4)}}\right) + \log\left(\frac{s-4}{s+4}\right)
\end{aligned} \tag{68}$$

\Rightarrow

$$-\log\left(\frac{\zeta(s+4)}{\zeta(-3-s)}\right) + \log\left(\frac{\zeta(-s+4)}{\zeta(-3+s)}\right) + \log(\pi^{2s}) = -\log\left(\frac{\zeta(s-4)}{\zeta(5-s)}\right) + \log\left(\frac{\zeta(-s-4)}{\zeta(5+s)}\right) + \log(\pi^{2s}) + \log\left(\frac{s-4}{s+4}\right) \tag{69}$$

\Rightarrow

$$-\log\left(\frac{\zeta(s+4)}{\zeta(-3-s)}\right) + \log\left(\frac{\zeta(-s+4)}{\zeta(-3+s)}\right) = -\log\left(\frac{\zeta(s-4)}{\zeta(5-s)}\right) + \log\left(\frac{\zeta(-s-4)}{\zeta(5+s)}\right) + \log\left(\frac{s-4}{s+4}\right). \tag{70}$$

This last statement is not difficult to justify. To this aim, we use the recursion relation substitutes as demonstrated:

$$\begin{aligned}
& -\log\left(\frac{\zeta(s+4)}{\zeta(-3-s)}\right) + \log\left(\frac{\zeta(-s+4)}{\zeta(-3+s)}\right) = -\log\left(\frac{\zeta(s-4)}{\zeta(5-s)}\right) + \log\left(\frac{\zeta(-s-4)}{\zeta(5+s)}\right) + \log\left(\frac{s-4}{s+4}\right) \\
& = -\log\left(\frac{(s-4)(s-3)(s-2)(s-1)(s)(s+1)(s+2)(s+3)}{(2\pi i)^4} \frac{\zeta(s+4)}{\zeta(-3-s)}\right) \\
& + \log\left(\frac{(-s-4)(-s-3)(-s-2)(-s-1)(-s)(-s+1)(-s+2)(-s+3)}{(2\pi i)^4} \frac{\zeta(-s+4)}{\zeta(-3+s)}\right) \\
& + \log\left(\frac{s-4}{s+4}\right)
\end{aligned} \tag{71}$$

which then confirms the obvious (but almost unreported) identity

$$\begin{aligned}
& \log\left(\frac{(s-4)(s-3)(s-2)(s-1)(s)(s+1)(s+2)(s+3)}{(2\pi i)^4}\right) \\
& - \log\left(\frac{(-s-4)(-s-3)(-s-2)(-s-1)(-s)(-s+1)(-s+2)(-s+3)}{(2\pi i)^4}\right) \\
& = \log\left(\frac{s-4}{s+4}\right).
\end{aligned} \tag{72}$$

presented here for the first time ever.

4.2 Logarithm of zeta at $\pm s$ and $1 \mp s$

Another important discovery involving eq. 70 is on the proof of the next theorem.

Theorem 4.2.

$$\begin{aligned}
& \log\left(\frac{\zeta(s)}{\zeta(1-s)}\right) + \log\left(\frac{\zeta(-s)}{\zeta(1+s)}\right) \\
&= \log\left(\frac{\zeta(-s+8)}{\zeta(-7+s)}\right) + \log\left(\frac{\zeta(s-8)}{\zeta(9-s)}\right) - \log\left(\frac{s-8}{s}\right) \\
&= \log\left(\frac{\zeta(-s+16)}{\zeta(-15+s)}\right) + \log\left(\frac{\zeta(s-16)}{\zeta(17-s)}\right) - \log\left(\frac{s-16}{s-8}\right) + \log\left(\frac{-s}{-s+8}\right) \\
&= \log\left(\frac{\zeta(s+8)}{\zeta(-7-s)}\right) + \log\left(\frac{\zeta(-s-8)}{\zeta(9+s)}\right) - \log\left(\frac{-s-8}{-s}\right) \\
&= \log\left(\frac{\zeta(s+16)}{\zeta(-15-s)}\right) + \log\left(\frac{\zeta(-s-16)}{\zeta(17+s)}\right) - \log\left(\frac{-s-16}{-s-8}\right) + \log\left(\frac{s}{s+8}\right)
\end{aligned} \tag{73}$$

Proof. The fact that

$$-\log\left(\frac{\zeta(s+4)}{\zeta(-3-s)}\right) + \log\left(\frac{\zeta(-s+4)}{\zeta(-3+s)}\right) = -\log\left(\frac{\zeta(s-4)}{\zeta(5-s)}\right) + \log\left(\frac{\zeta(-s-4)}{\zeta(5+s)}\right) + \log\left(\frac{s-4}{s+4}\right)$$

suggests

$$-\log\left(\frac{\zeta(s)}{\zeta(1-s)}\right) + \log\left(\frac{\zeta(-s+8)}{\zeta(-7+s)}\right) = -\log\left(\frac{\zeta(s-8)}{\zeta(9-s)}\right) + \log\left(\frac{\zeta(-s)}{\zeta(1+s)}\right) + \log\left(\frac{s-8}{s}\right) \tag{74}$$

\Rightarrow

$$-\log\left(\frac{\zeta(-s)}{\zeta(1+s)}\right) - \log\left(\frac{\zeta(s)}{\zeta(1-s)}\right) = -\log\left(\frac{\zeta(-s+8)}{\zeta(-7+s)}\right) - \log\left(\frac{\zeta(s-8)}{\zeta(9-s)}\right) + \log\left(\frac{s-8}{s}\right) \tag{75}$$

\Rightarrow

$$\log\left(\frac{\zeta(-s)}{\zeta(1+s)}\right) + \log\left(\frac{\zeta(s)}{\zeta(1-s)}\right) = \log\left(\frac{\zeta(-s+8)}{\zeta(-7+s)}\right) + \log\left(\frac{\zeta(s-8)}{\zeta(9-s)}\right) - \log\left(\frac{s-8}{s}\right) \tag{76}$$

This implies

$$\begin{aligned}
\log\left(\frac{\zeta(s)}{\zeta(1-s)}\right) + \log\left(\frac{\zeta(-s)}{\zeta(1+s)}\right) &= \log\left(\frac{\zeta(-s+8)}{\zeta(-7+s)}\right) + \log\left(\frac{\zeta(s-8)}{\zeta(9-s)}\right) - \log\left(\frac{s-8}{s}\right) \\
&= \log\left(\frac{\zeta(s+8)}{\zeta(-7-s)}\right) + \log\left(\frac{\zeta(-s-8)}{\zeta(9+s)}\right) - \log\left(\frac{-s-8}{-s}\right).
\end{aligned} \tag{77}$$

Substituting $s \rightarrow -s+8$ into 77:

$$\begin{aligned}
\log\left(\frac{\zeta(-s+8)}{\zeta(-7+s)}\right) + \log\left(\frac{\zeta(s-8)}{\zeta(9-s)}\right) &= \log\left(\frac{\zeta(s)}{\zeta(1-s)}\right) + \log\left(\frac{\zeta(-s)}{\zeta(1+s)}\right) - \log\left(\frac{-s}{-s+8}\right) \\
&= \log\left(\frac{\zeta(-s+16)}{\zeta(-15+s)}\right) + \log\left(\frac{\zeta(s-16)}{\zeta(17-s)}\right) - \log\left(\frac{s-16}{s-8}\right).
\end{aligned} \tag{78}$$

Substituting $s \rightarrow s+8$ into 77:

$$\begin{aligned}
\log\left(\frac{\zeta(s+8)}{\zeta(-7-s)}\right) + \log\left(\frac{\zeta(-s-8)}{\zeta(9+s)}\right) &= \log\left(\frac{\zeta(-s)}{\zeta(1+s)}\right) + \log\left(\frac{\zeta(s)}{\zeta(1-s)}\right) - \log\left(\frac{s}{s+8}\right) \\
&= \log\left(\frac{\zeta(s+16)}{\zeta(-15-s)}\right) + \log\left(\frac{\zeta(-s-16)}{\zeta(17+s)}\right) - \log\left(\frac{-s-16}{-s-8}\right).
\end{aligned} \tag{79}$$

In other words,

$$\begin{aligned}
& \log\left(\frac{\zeta(s)}{\zeta(1-s)}\right) + \log\left(\frac{\zeta(-s)}{\zeta(1+s)}\right) \\
&= \log\left(\frac{\zeta(-s+8)}{\zeta(-7+s)}\right) + \log\left(\frac{\zeta(s-8)}{\zeta(9-s)}\right) - \log\left(\frac{s-8}{s}\right) \\
&= \log\left(\frac{\zeta(-s+16)}{\zeta(-15+s)}\right) + \log\left(\frac{\zeta(s-16)}{\zeta(17-s)}\right) - \log\left(\frac{s-16}{s-8}\right) + \log\left(\frac{-s}{-s+8}\right) \\
&= \log\left(\frac{\zeta(s+8)}{\zeta(-7-s)}\right) + \log\left(\frac{\zeta(-s-8)}{\zeta(9+s)}\right) - \log\left(\frac{-s-8}{-s}\right) \\
&= \log\left(\frac{\zeta(s+16)}{\zeta(-15-s)}\right) + \log\left(\frac{\zeta(-s-16)}{\zeta(17+s)}\right) - \log\left(\frac{-s-16}{-s-8}\right) + \log\left(\frac{s}{s+8}\right)
\end{aligned}$$

□

5 An introduction to the new omega function $\Omega(s)$

Here, we wish to present and prove

Theorem 5.1.

$$\begin{aligned}
\Omega(s) &= \psi^{(2n)}\left(\frac{s}{2}\right) + \psi^{(2n)}\left(\frac{1-s}{2}\right) - \psi^{(2n)}\left(1 - \frac{s}{2}\right) - \psi^{(2n)}\left(\frac{1+s}{2}\right) = \\
&= \left(\psi^{(2n)}\left(\frac{s}{2}\right) - \psi^{(2n)}\left(\frac{1}{2} + \frac{s}{2}\right)\right) + \left(\psi^{(2n)}\left(\frac{1-s}{2}\right) - \psi^{(2n)}\left(\frac{1}{2} + \frac{1-s}{2}\right)\right) = \\
&= (-2)^{2n+1} \Gamma(2n+1) \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+s)^{2n+1}} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1-s)^{2n+1}} \right) = \\
&= 2^{2n} \frac{d^{2n+1}}{ds^{2n+1}} \left(-\log\left(\frac{\sin(\frac{\pi(\frac{s+2}{2})}{2})}{\cos(\frac{\pi(\frac{s+1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\cos(\frac{\pi(\frac{s}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s}{2})}{2})}{\cos(\frac{\pi(\frac{s-1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\cos(\frac{\pi(\frac{s-2}{2})}{2})}\right) \right) = \\
&= -\Gamma(2n+1) \left(\sum_{k=0}^{\infty} \frac{1}{(k+\frac{s}{2})^{2n+1}} + \sum_{k=0}^{\infty} \frac{1}{(k+\frac{1-s}{2})^{2n+1}} - \sum_{k=0}^{\infty} \frac{1}{(k+1-\frac{s}{2})^{2n+1}} - \sum_{k=0}^{\infty} \frac{1}{(k+\frac{1+s}{2})^{2n+1}} \right)
\end{aligned} \tag{80}$$

Proof. According to theorem 3.4:

$$\begin{aligned}
& \log\left(\frac{\Gamma^2(-\frac{s}{2})}{\Gamma^2(\frac{1+s}{2})}\right) + \log(\pi^{2(1+s-\frac{1}{2})}) = -\log\left(\frac{\Gamma^2(\frac{s}{2})}{\Gamma^2(\frac{1-s}{2})}\right) - \log(\pi^{2(1-s-\frac{1}{2})}) - \log\left(\frac{2s^2}{(s-2)} \frac{(\frac{s}{2}-1)}{(2\pi i)^2}\right) + \\
& - \log\left(\frac{\sin(\frac{\pi(\frac{s+2}{2})}{2})}{\sin(\frac{\pi(1-\frac{s+1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right) \\
& \Rightarrow \\
& \frac{d^{2n+1}}{ds^{2n+1}} \left(\log\left(\frac{\Gamma^2(-\frac{s}{2})}{\Gamma^2(\frac{1+s}{2})}\right) \right) = -\frac{d^{2n+1}}{ds^{2n+1}} \left(\log\left(\frac{\Gamma^2(\frac{s}{2})}{\Gamma^2(\frac{1-s}{2})}\right) \right) - \frac{d^{2n+1}}{ds^{2n+1}} \left(\log\left(\frac{2s^2}{(s-2)} \frac{(\frac{s}{2}-1)}{(2\pi i)^2}\right) \right) + \\
& \frac{d^{2n+1}}{ds^{2n+1}} \left(-\log\left(\frac{\sin(\frac{\pi(\frac{s+2}{2})}{2})}{\sin(\frac{\pi(1-\frac{s+1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right) \right) \\
& \Rightarrow
\end{aligned} \tag{81}$$

$$\begin{aligned}
& -\frac{\psi^{(2)}(-\frac{s}{2}) + \psi^{(2)}(\frac{1+s}{2})}{2^{2n}} = -\frac{\psi^{(2)}(\frac{s}{2}) + \psi^{(2)}(\frac{1-s}{2})}{2^{2n}} - \frac{d^{2n+1}}{ds^{2n+1}}(\log(s^2)) + \\
& \frac{d^{2n+1}}{ds^{2n+1}} \left(-\log\left(\frac{\sin(\frac{\pi(\frac{s+2}{2})}{2})}{\sin(\frac{\pi(1-\frac{s+1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right) \right)
\end{aligned} \tag{82}$$

Note the following identities:

$$\psi^{(2n)}\left(\frac{a}{b}\right) = \frac{(-1)^{2n+1}\Gamma(2n+1)}{(\frac{a}{b})^{2n+1}} + \psi^{(2n)}\left(1 + \frac{a}{b}\right);$$

$$\psi^{(2n)}\left(1 + \frac{a}{b}\right) = \frac{(-1)^{2n}\Gamma(2n+1)}{(\frac{a}{b})^{2n+1}} + \psi^{(2n)}\left(\frac{a}{b}\right);$$

i.e.

$$\psi^{(2n)}\left(-\frac{s}{2}\right) = -\frac{\Gamma(2n+1)}{(-\frac{s}{2})^{2n+1}} + \psi^{(2n)}\left(1 - \frac{s}{2}\right).$$

Therefore

$$\begin{aligned}
& -\frac{\psi^{(2)}(1 - \frac{s}{2}) + \psi^{(2)}(\frac{1+s}{2})}{2^{2n}} = -\frac{\psi^{(2)}(\frac{s}{2}) + \psi^{(2)}(\frac{1-s}{2})}{2^{2n}} - \frac{\Gamma(2n+1)}{2^{2n}(-\frac{s}{2})^{2n+1}} - \frac{d^{2n+1}}{ds^{2n+1}}(\log(s^2)) + \\
& \frac{d^{2n+1}}{ds^{2n+1}} \left(-\log\left(\frac{\sin(\frac{\pi(\frac{s+2}{2})}{2})}{\sin(\frac{\pi(1-\frac{s+1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right) \right)
\end{aligned} \tag{83}$$

\Rightarrow

$$\begin{aligned}
& -\frac{\psi^{(2)}(1 - \frac{s}{2}) + \psi^{(2)}(\frac{1+s}{2})}{2^{2n}} = -\frac{\psi^{(2)}(\frac{s}{2}) + \psi^{(2)}(\frac{1-s}{2})}{2^{2n}} + \frac{2\Gamma(2n+1)}{s^{2n+1}} - \frac{d^{2n+1}}{ds^{2n+1}}(\log(s^2)) + \\
& \frac{d^{2n+1}}{ds^{2n+1}} \left(-\log\left(\frac{\sin(\frac{\pi(\frac{s+2}{2})}{2})}{\sin(\frac{\pi(1-\frac{s+1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right) \right)
\end{aligned} \tag{84}$$

\Rightarrow

$$\begin{aligned}
& -\frac{\psi^{(2)}(1 - \frac{s}{2}) + \psi^{(2)}(\frac{1+s}{2})}{2^{2n}} = -\frac{\psi^{(2)}(\frac{s}{2}) + \psi^{(2)}(\frac{1-s}{2})}{2^{2n}} + \\
& + \frac{d^{2n+1}}{ds^{2n+1}} \left(-\log\left(\frac{\sin(\frac{\pi(\frac{s+2}{2})}{2})}{\sin(\frac{\pi(1-\frac{s+1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right) \right)
\end{aligned} \tag{85}$$

\Rightarrow

$$\begin{aligned}
& \frac{\psi^{(2)}(\frac{s}{2}) + \psi^{(2)}(\frac{1-s}{2})}{2^{2n}} - \frac{\psi^{(2)}(1 - \frac{s}{2}) + \psi^{(2)}(\frac{1+s}{2})}{2^{2n}} = \\
& \frac{d^{2n+1}}{ds^{2n+1}} \left(-\log\left(\frac{\sin(\frac{\pi(\frac{s+2}{2})}{2})}{\sin(\frac{\pi(1-\frac{s+1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\sin(\frac{\pi(1-\frac{s-2}{2})}{2})}\right) \right) = \\
& -\frac{\Gamma(2n+1)}{2^{2n}} \left(\sum_{k=0}^{\infty} \frac{1}{(k + \frac{s}{2})^{2n+1}} + \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1-s}{2})^{2n+1}} - \sum_{k=0}^{\infty} \frac{1}{(k + 1 - \frac{s}{2})^{2n+1}} - \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1+s}{2})^{2n+1}} \right)
\end{aligned} \tag{86}$$

This leads us to establish and define the new function Omega $\Omega(s)$ as

$$\begin{aligned}\Omega(s) &= \psi^{(2)}\left(\frac{s}{2}\right) + \psi^{(2)}\left(\frac{1-s}{2}\right) - \psi^{(2)}\left(1 - \frac{s}{2}\right) - \psi^{(2)}\left(\frac{1+s}{2}\right) = \\ 2^{2n} \frac{d^{2n+1}}{ds^{2n+1}} &\left(-\log\left(\frac{\sin\left(\frac{\pi(\frac{s+2}{2})}{2}\right)}{\sin\left(\frac{\pi(1-\frac{s+1}{2})}{2}\right)}\right) + \log\left(\frac{\sin\left(\frac{\pi(\frac{s+1}{2})}{2}\right)}{\sin\left(\frac{\pi(1-\frac{s}{2})}{2}\right)}\right) - \log\left(\frac{\sin\left(\frac{\pi(\frac{s}{2})}{2}\right)}{\sin\left(\frac{\pi(1-\frac{s-1}{2})}{2}\right)}\right) + \log\left(\frac{\sin\left(\frac{\pi(\frac{s-1}{2})}{2}\right)}{\sin\left(\frac{\pi(1-\frac{s-2}{2})}{2}\right)}\right) \right) = \\ -\Gamma(2n+1) &\left(\sum_{k=0}^{\infty} \frac{1}{(k + \frac{s}{2})^{2n+1}} + \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1-s}{2})^{2n+1}} - \sum_{k=0}^{\infty} \frac{1}{(k + 1 - \frac{s}{2})^{2n+1}} - \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1+s}{2})^{2n+1}} \right).\end{aligned}\quad (87)$$

such that

$$\begin{aligned}\Omega(s) &= \psi^{(2n)}\left(\frac{s}{2}\right) + \psi^{(2n)}\left(\frac{1-s}{2}\right) - \psi^{(2n)}\left(1 - \frac{s}{2}\right) - \psi^{(2n)}\left(\frac{1+s}{2}\right) = \\ &\left(\psi^{(2n)}\left(\frac{s}{2}\right) - \psi^{(2n)}\left(\frac{1}{2} + \frac{s}{2}\right) \right) + \left(\psi^{(2n)}\left(\frac{1-s}{2}\right) - \psi^{(2n)}\left(\frac{1}{2} + \frac{1-s}{2}\right) \right) = \\ &(-2)^{2n+1} \Gamma(2n+1) \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+s)^{2n+1}} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1-s)^{2n+1}} \right) = \\ 2^{2n} \frac{d^{2n+1}}{ds^{2n+1}} &\left(-\log\left(\frac{\sin\left(\frac{\pi(\frac{s+2}{2})}{2}\right)}{\cos\left(\frac{\pi(\frac{s+1}{2})}{2}\right)}\right) + \log\left(\frac{\sin\left(\frac{\pi(\frac{s+1}{2})}{2}\right)}{\cos\left(\frac{\pi(\frac{s}{2})}{2}\right)}\right) - \log\left(\frac{\sin\left(\frac{\pi(\frac{s}{2})}{2}\right)}{\cos\left(\frac{\pi(\frac{s-1}{2})}{2}\right)}\right) + \log\left(\frac{\sin\left(\frac{\pi(\frac{s-1}{2})}{2}\right)}{\cos\left(\frac{\pi(\frac{s-2}{2})}{2}\right)}\right) \right) = \\ -\Gamma(2n+1) &\left(\sum_{k=0}^{\infty} \frac{1}{(k + \frac{s}{2})^{2n+1}} + \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1-s}{2})^{2n+1}} - \sum_{k=0}^{\infty} \frac{1}{(k + 1 - \frac{s}{2})^{2n+1}} - \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1+s}{2})^{2n+1}} \right)\end{aligned}\quad (88)$$

□

The functional equations associated with this new function satisfy the following properties:

$$\begin{aligned}\Omega(s) &= \Omega(1-s); \\ \Omega(s) &= -\Omega(-s); \\ \Omega(s) &= -\Omega(s \pm 1); \\ \Omega(s) &= \Omega(s \pm 2).\end{aligned}\quad (89)$$

As a result the following fundamental properties emerge in general:

$$\begin{aligned}\Omega(s) &= -\Omega(s \pm 2j + 1) = \Omega(-s \pm 2j + 1); \\ \Omega(s) &= \Omega(s \pm 2j) = -\Omega(-s \pm 2j);\end{aligned}\quad (90)$$

where j is any integer number. This omega function has poles at integer points. For instance, at the point $s=0$ and $s=1$, the omega function, i.e. $\Omega(0)$ or $\Omega(1)$, requires the estimation of $\sum_{k=0}^{\infty} \frac{1}{k^{2n+1}} = \frac{1}{0} + \zeta(2n+1)$:

$$\Omega(0) = \Omega(1) = -\Gamma(2n+1) \left(\sum_{k=0}^{\infty} \frac{1}{k^{2n+1}} + \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1}{2})^{2n+1}} - \sum_{k=0}^{\infty} \frac{1}{(k+1)^{2n+1}} - \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1}{2})^{2n+1}} \right). \quad (91)$$

5.1 The omega function of golden ratio

Let ϕ represent the golden ratio constant $\frac{1+\sqrt{5}}{2}$. The implication of the omega functional equation is that $\Omega(\phi) = \Omega(1-\phi) = \Omega(1 + \frac{1}{\phi}) = \Omega(-\frac{1}{\phi})$ because $\phi = 1 + (\phi - 1) = 1 + \frac{1}{\phi}$. This suggests that all the series sums below will produce the same result.

$s \rightarrow \phi$:

$$\frac{\Omega(\phi)}{-\Gamma(2n+1)} = \sum_{k=0}^{\infty} \frac{1}{(k + \frac{\phi}{2})^{2n+1}} + \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1}{2} - \frac{\phi}{2})^{2n+1}} - \sum_{k=0}^{\infty} \frac{1}{(k + 1 - \frac{\phi}{2})^{2n+1}} - \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1}{2} + \frac{\phi}{2})^{2n+1}}. \quad (92)$$

$s \rightarrow 1 - \phi$:

$$\frac{\Omega(1 - \phi)}{-\Gamma(2n+1)} = \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1}{2} - \frac{\phi}{2})^{2n+1}} + \sum_{k=0}^{\infty} \frac{1}{(k + \frac{\phi}{2})^{2n+1}} - \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1}{2} + \frac{\phi}{2})^{2n+1}} - \sum_{k=0}^{\infty} \frac{1}{(k + 1 - \frac{\phi}{2})^{2n+1}}. \quad (93)$$

$s \rightarrow 1 + \frac{1}{\phi}$:

$$\begin{aligned} \frac{\Omega(1 + \frac{1}{\phi})}{-\Gamma(2n+1)} &= \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1}{2} + \frac{1}{2\phi})^{2n+1}} + \sum_{k=0}^{\infty} \frac{1}{(k - \frac{1}{2\phi})^{2n+1}} - \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1}{2} - \frac{1}{2\phi})^{2n+1}} - \sum_{k=0}^{\infty} \frac{1}{(k + 1 + \frac{1}{2\phi})^{2n+1}} \\ \frac{\Omega(1 + \frac{1}{\phi})}{-\Gamma(2n+1)} &= \sum_{k=0}^{\infty} \frac{1}{(k - \frac{1}{2\phi})^{2n+1}} - \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1}{2} - \frac{1}{2\phi})^{2n+1}} - \sum_{k=0}^{\infty} \frac{1}{(k + 1 + \frac{1}{2\phi})^{2n+1}} + \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1}{2} + \frac{1}{2\phi})^{2n+1}}. \end{aligned} \quad (94)$$

$s \rightarrow -\frac{1}{\phi}$:

$$\frac{\Omega(-\frac{1}{\phi})}{-\Gamma(2n+1)} = \sum_{k=0}^{\infty} \frac{1}{(k - \frac{1}{2\phi})^{2n+1}} + \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1}{2} + \frac{1}{2\phi})^{2n+1}} - \sum_{k=0}^{\infty} \frac{1}{(k + 1 + \frac{1}{2\phi})^{2n+1}} - \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1}{2} - \frac{1}{2\phi})^{2n+1}}. \quad (95)$$

6 Discussion

In this paper we derived a new formula for representing the actual value of $\psi^{2n}(s)$ from which we further derived another formula for expressing the value of $\zeta(2n+1)$ employing a method dependent on $\psi^{2n}(s) + \psi^{2n}(1-s)$ calculation. For instance, we uncovered the new formula for the first time

$$\zeta(2n+1) = -\frac{(\psi^{(2n)}(\frac{1}{4}) + \psi^{(2n)}(\frac{3}{4}))}{2^{2n+1} \cdot (2^{2n+1} - 1)} \cdot \frac{1}{\Gamma(2n+1)} = -\frac{2 \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\frac{\zeta(1-s)}{\zeta(s)})) \Big|_{s \rightarrow \frac{1}{4}} + \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\tan(\frac{\pi s}{2}))) \Big|_{s \rightarrow \frac{1}{4}}}{2^{2n+1}(2^{2n+1} - 1)\Gamma(2n+1)}$$

We then presented new strategies for calculating the value of the logarithm of $\frac{\zeta(s)}{\zeta(1-s)}$ based on fractions of other zeta functions at smaller points of arguments. New identities relating the logarithm of Riemann zeta function to that of gamma function were also presented and all proved primarily using elementary functions only. We considered presenting new combinatorial results and perspectives on contemporay zeta functional analysis to establish deeper connections with several related zeta functions for the first time. We then used one of our main results about a recent discovery (also presented here) to uncover and present the relation

$$\begin{aligned}
\Omega(s) &= \psi^{(2n)}\left(\frac{s}{2}\right) + \psi^{(2n)}\left(\frac{1-s}{2}\right) - \psi^{(2n)}\left(1 - \frac{s}{2}\right) - \psi^{(2n)}\left(\frac{1+s}{2}\right) = \\
&= \left(\psi^{(2n)}\left(\frac{s}{2}\right) - \psi^{(2n)}\left(\frac{1}{2} + \frac{s}{2}\right)\right) + \left(\psi^{(2n)}\left(\frac{1-s}{2}\right) - \psi^{(2n)}\left(\frac{1}{2} + \frac{1-s}{2}\right)\right) = \\
&= (-2)^{2n+1}\Gamma(2n+1) \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+s)^{2n+1}} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1-s)^{2n+1}} \right) = \\
&= 2^{2n} \frac{d^{2n+1}}{ds^{2n+1}} \left(-\log\left(\frac{\sin(\frac{\pi(\frac{s+2}{2})}{2})}{\cos(\frac{\pi(\frac{s+1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\cos(\frac{\pi(\frac{s}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s}{2})}{2})}{\cos(\frac{\pi(\frac{s-1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\cos(\frac{\pi(\frac{s-2}{2})}{2})}\right) \right) = \\
&= -\Gamma(2n+1) \left(\sum_{k=0}^{\infty} \frac{1}{(k+\frac{s}{2})^{2n+1}} + \sum_{k=0}^{\infty} \frac{1}{(k+\frac{1-s}{2})^{2n+1}} - \sum_{k=0}^{\infty} \frac{1}{(k+1-\frac{s}{2})^{2n+1}} - \sum_{k=0}^{\infty} \frac{1}{(k+\frac{1+s}{2})^{2n+1}} \right)
\end{aligned}$$

which may provide new optimised and efficient strategies for analysing and evaluating sums of alternating series related to zeta, Dirichlet beta, and important nonelementary functions. For example,

$$\begin{aligned}
-2(2^{2n+1})2^{2n+1}\beta(2n+1) &= \Omega\left(\frac{1}{4}\right) = 2\psi^{(2n)}\left(\frac{1}{4}\right) - 2\psi^{(2n)}\left(\frac{3}{4}\right) = \\
&= (-2)^{2n+1}\Gamma(2n+1) \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+\frac{1}{2})^{2n+1}} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+\frac{1}{2})^{2n+1}} \right) = \quad (96) \\
&= 2^{2n} \frac{d^{2n+1}}{ds^{2n+1}} \left(-\log\left(\frac{\sin(\frac{\pi(\frac{s+2}{2})}{2})}{\cos(\frac{\pi(\frac{s+1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\cos(\frac{\pi(\frac{s}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s}{2})}{2})}{\cos(\frac{\pi(\frac{s-1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\cos(\frac{\pi(\frac{s-2}{2})}{2})}\right) \right) \Big|_{x \rightarrow \frac{1}{2}}
\end{aligned}$$

It is a delight knowing that the same technique could be used to analyse slightly more complicated results such as

$$\begin{aligned}
\Omega(i) &= \psi^{(2n)}\left(\frac{i}{2}\right) + \psi^{(2n)}\left(\frac{1-i}{2}\right) - \psi^{(2n)}\left(1 - \frac{i}{2}\right) - \psi^{(2n)}\left(\frac{1+i}{2}\right) = \\
&= (-2)^{2n+1}\Gamma(2n+1) \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+i)^{2n+1}} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1-i)^{2n+1}} \right) = \\
&= (-2)^{2n+1}\Gamma(2n+1) \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+i)^{2n+1}} - \sum_{k=1}^{\infty} \frac{(-1)^k}{(k-i)^{2n+1}} \right) = \quad (97) \\
&= (-2)^{2n+1}\Gamma(2n+1) \left(\frac{1}{i^{2n+1}} + \sum_{k=1}^{\infty} \frac{(-1)^k((k-i)^{2n+1} - (k+i)^{2n+1})}{(k^2+1)^{2n+1}} \right) = \\
&= 2^{2n} \frac{d^{2n+1}}{ds^{2n+1}} \left(-\log\left(\frac{\sin(\frac{\pi(\frac{s+2}{2})}{2})}{\cos(\frac{\pi(\frac{s+1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s+1}{2})}{2})}{\cos(\frac{\pi(\frac{s}{2})}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(\frac{s}{2})}{2})}{\cos(\frac{\pi(\frac{s-1}{2})}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(\frac{s-1}{2})}{2})}{\cos(\frac{\pi(\frac{s-2}{2})}{2})}\right) \right) \Big|_{x \rightarrow i}
\end{aligned}$$

which has instantaneously provided a stable mechanism of evaluating the series sum in terms of derivatives of the logarithm of elementary trigonometric functions; as a result, avoiding actual computations of $\psi^{(2n)}(\frac{i}{2})$, $\psi^{(2n)}(\frac{1-i}{2})$, $\psi^{(2n)}(1 - \frac{i}{2})$ and $\psi^{(2n)}(\frac{1+i}{2})$ without requiring factorising $((k-i)^{2n+1} - (k+i)^{2n+1})$.

We then ask: how important is the result

$$\begin{aligned} & \frac{1}{i^{2n+1}} + \sum_{k=1}^{\infty} \frac{(-1)^k ((k-i)^{2n+1} - (k+i)^{2n+1})}{(k^2+1)^{2n+1}} \\ &= - \frac{\frac{d^{2n+1}}{ds^{2n+1}} \left(-\log\left(\frac{\sin(\frac{\pi(s+2)}{2})}{\cos(\frac{\pi(s+1)}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(s+1)}{2})}{\cos(\frac{\pi(s)}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(s)}{2})}{\cos(\frac{\pi(s-1)}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(s-1)}{2})}{\cos(\frac{\pi(s-2)}{2})}\right) \right) \Big|_{x \rightarrow i}}{2\Gamma(2n+1)} \end{aligned} \quad (98)$$

We ponder on such interesting questions and sense the need to review the result

$$\begin{aligned} & \frac{1}{(-6i)i^3} + \sum_{k=1}^{\infty} \frac{(-1)^k (k^2 - \frac{1}{3})}{(k^2+1)^3} \\ &= - \frac{\frac{d^3}{ds^3} \left(-\log\left(\frac{\sin(\frac{\pi(s+2)}{2})}{\cos(\frac{\pi(s+1)}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(s+1)}{2})}{\cos(\frac{\pi(s)}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(s)}{2})}{\cos(\frac{\pi(s-1)}{2})}\right) + \log\left(\frac{\sin(\frac{\pi(s-1)}{2})}{\cos(\frac{\pi(s-2)}{2})}\right) \right) \Big|_{x \rightarrow i}}{2(-6i)\Gamma(2n+1)}. \end{aligned} \quad (99)$$

which was derived after just a few additional steps.

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